MATH 120A Prep: Functions

Facts to Know:

Function Properties: Consider a function $f : X \to Y$.

- Injective/One-to-one - Each input goes to a diffrent output.

To prove: Assume $f(x_1) = f(x_2)$, show $x_1 = x_2$.

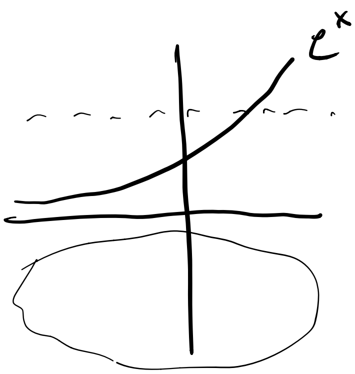
- Surjective/Onto - Every element in the codomain (Y) is the image of an element of X.

To prove: let $y \in Y$, want to show there is an $x \in X$ such that $f(x) = y$.

- Bijective - A function is bijective if and only if it is injective <u>and</u> surjective.

Examples:

1. (a) Determine whether the exponential map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$ is injective and/or surjective.



Injective: Suppose $f(x_1) = f(x_2)$, so $e^{x_1} = e^{x_2}$.

Apply ln to both sides, so $\ln(e^{x_1}) = \ln(e^{x_2})$

$x_1 = x_2$. ✓ e^x is injective.

Surjective: e^x is never negative or zero, so it is <u>not</u> surjective.

(b) What changes if we consider this as a function $f : \mathbb{R} \to \mathbb{R}^+$ where $\mathbb{R}^+ = \{r \in \mathbb{R} : r > 0\}$?

<u>Injective</u>: Same proof applies.

Surjective: let $r \in \mathbb{R}^+$, so $r > 0$. Want to find is a number x in $\mathbb{R}$ so $e^x = r$. Let $x = \ln(r)$.

$e^x = e^{\ln(r)} = r$, so <u>surjective</u>.

This function is a bijection.

2. Is the map $g : \mathbb{R}^2 \to \mathbb{R}$ where $g(x, y) = x^2 - y^2$ injective? Is it surjective?

Injective: No. $f(x) = x^2$ is not injective since $1^2 = 1$, $(-1)^2 = 1$.

Choose $y = 0$, $g(x,0) = x^2$

$$\left.\begin{aligned} g(1,0) &= 1^2 - 0^2 = 1 \\ g(-1,0) &= (-1)^2 - 0^2 = 1 \end{aligned}\right\} \text{ not injective.}$$

Surjective: x^2 is positive

$-y^2$ is negative.

$g(x,y)$ is surjective

Case 1: $r < 0$

$-r > 0$, so we can get $\sqrt{-r}$

$$g(0, \sqrt{-r}) = 0^2 - (\sqrt{-r})^2$$

$$= -(-r) = r \checkmark$$

Case 2: $r \geq 0$

Consider $\sqrt{r}$

$$g(\sqrt{r}, 0) = \sqrt{r}^2 - 0^2$$

$$= r - 0$$

$$= r \checkmark$$

3. Let S be the set $\{(x, y) \in \mathbb{R}^2 : x \neq y\}$. Show the map $h : S \to \mathbb{R}^2$ defined by $h(x, y) = (x - y, x^2 - y^2)$ is injective but not surjective.

Injective: Suppose we have (x_1, y_1) and (x_2, y_2) such that

$h(x_1, y_1) = h(x_2, y_2) \to (x_1 - y_1, x_1^2 - y_1^2) = (x_2 - y_2, x_2^2 - y_2^2)$

$0 \neq x_1 - y_1 = x_2 - y_2 \neq 0$

$x_1^2 - y_1^2 = x_2^2 - y_2^2 \longrightarrow \cancel{(x_1 - y_1)}(x_1 + y_1) = \cancel{(x_2 - y_2)}(x_2 + y_2)$

$$\begin{aligned} & x_1 + y_1 = x_2 + y_2 \\ + \quad & x_1 - y_1 = x_2 - y_2 \\ \hline & 2x_1 = 2x_2 \end{aligned}$$

$\to y_1 = y_2$ $\quad (x_1, y_1) = (x_2, y_2)$

so injective $\checkmark$

$2x_1 = 2x_2 \to x_1 = x_2$

Not surjective: $h(x,y) = (0,0)$ $\quad h(x,y) = (x-y, x^2 - y^2) = (0,0)$

$\to x - y = 0 \longrightarrow x = y$ so not in the domain.